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Influence of variable properties on the stability of two-dimensional boundary layers

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Classical linear stability theory is extended to include the effects of temperature- and pressure-dependent fluid properties. These effects are studied asymptotically by using Taylor series expansions for all the properties with respect to temperature and pressure. In this asymptotic approach all effects are well separated from each other, and only the Prandtl number remains as a parameter. In their general form the asymptotic solutions hold for all Newtonian fluids. A shooting technique with Gram–Schmidt orthonormalization for the zero-order equation (classical Orr–Sommerfeld problem) and a multiple shooting method for all other equations is applied to solve the stiff differential equations. In particular the zero- and first-order equations are solved for a flat-plate boundary-layer flow with temperature-dependent viscosity. Physically, this corresponds to a fluid with a linear viscosity/temperature relation. The results show that decreasing the viscosity in the near-wall region of the boundary layer stabilizes the flow, whereas it would be destabilized for a uniformly decreased viscosity.

1. Introduction

Among the studies that have investigated the stability of laminar boundary-layer flows, only a few have taken into account the effect of variable properties, even though non-constant properties can have a strong effect on the critical Reynolds number. For example, Wazzan *et al.* (1972) investigated the boundary-layer stability of water under non-isothermal conditions. They found that the critical Reynolds number for a heated flat-plate boundary layer in water varies between 520 and nearly 16000. Other studies of forced-convection stability which take into account variable-property effects in a more or less systematic way are those by Hauptmann (1968), Lee, Chen & Armaly (1990) and Asfar, Masad & Nayfeh (1990). Natural-convection flows with variable property effects beyond that of the Boussinesq approximation were studied by Sabhapathy & Cheng (1986) and Chen & Pearlstein (1989), for example.

The present study outlines a general method which includes the effect of small temperature and pressure variations on the physical properties. Results of this analysis will hold for all Newtonian fluids instead of just one particular fluid. The basic approach starts from a Taylor series expansion of the properties with respect to temperature and pressure. Next, a regular perturbation is applied to the basic equations of stability with the constant-property case representing leading-order behaviour.

The basic stability equations which allow for the variation of all physical properties are given in §2. In §3, the regular perturbation procedure is described in detail. The method is then applied to the case of temperature-dependent viscosity in §4. Specific results based on the numerical solutions of the zero- and first-order equations are provided in §5 for the flat-plate boundary layer. In the discussion, §6, the general asymptotic results are compared to the results of Wazzan *et al.* (1972) by specifying the Prandtl number to match the particular fluid of that study.

2. Basic equations

For constant properties, the fundamental differential equation in the so-called method of small disturbances is the Orr–Sommerfeld (OS) equation, see for example Schlichting (1982). Squire (1933) has shown that it is sufficient to consider two-dimensional disturbances. For variable properties, the same result was found by Yih (1955) by extending the method of Squire.

The present analysis presents an extended version of the OS equation which holds when all physical properties vary. To keep it as general as possible the equation is non-dimensionalized with a reference length L^*_R and a reference velocity U^*_R, which are specified later. All dimensional quantities are starred, and all complex quantities are marked by the symbol $\hat{}$. In the method of small disturbances all quantities are decomposed into a mean value, $\bar{a}^*$, and a superimposed disturbance a'^* Here, a^* represents the velocity components u^* and v^* (two-dimensional flow) and the pressure p^*. When variable properties are involved, it also represents these properties, i.e. density ρ^*, viscosity μ^*, thermal conductivity k^* and specific heat c^*_p, as well as the temperature T^*. Owing to the temperature dependence of the properties, the modified OS equation must be supplemented by the thermal energy equation for the disturbance.

The common assumption (e.g. Schlichting 1982) is that any arbitrary two-dimensional disturbance can be expanded in a Fourier series; thus a single oscillation of the disturbance is assumed to be of the form (temporal stability)

$$\hat{a}'^*(x^*, y^*, t^*) = \hat{a}^*(y^*)\exp[\mathrm{i}\alpha^*(x^*-\hat{c}^*t^*)]. \tag{2.1}$$

In (2.1) α^* is real with $2\pi/\alpha^*$ being the wavelength of the disturbance. The quantity $\hat{c}^*$ is complex,

$$\hat{c}^* = c^*_r + \mathrm{i}c^*_i. \tag{2.2}$$

Here c^*_r denotes the phase velocity whereas c^*_i determines the degree of amplification or damping.

From the Navier–Stokes equations and the thermal energy equation (both for variable properties), together with the continuity equation, the following linearized differential equations for the dimensionless amplitude functions $\hat{u}(y)$, $\hat{v}(y)$ and $\hat{\theta}(y)$ are derived by inserting (2.1), subtracting the mean flow equations, and eliminating the pressure in the momentum equations:

$$\hat{\rho}(\bar{u}-\hat{c})+\bar{\rho}'\frac{\hat{v}}{\mathrm{i}\alpha}+\bar{\rho}\left[\hat{u}+\frac{\hat{v}'}{\mathrm{i}\alpha}\right]=0, \tag{2.3}$$

$$\bar{\rho}\left\{(\bar{u}-\hat{c})\left(\hat{u}'+\alpha^2\frac{\hat{v}}{\mathrm{i}\alpha}\right)+\bar{u}''\frac{\hat{v}}{\mathrm{i}\alpha}+\bar{u}'\left[\hat{u}+\frac{\hat{v}'}{\mathrm{i}\alpha}\right]\right\}$$
$$-\frac{1}{Fr^2}\left\{\hat{\rho}\cos\sigma+\frac{\mathrm{i}}{\alpha}\hat{\rho}'\sin\sigma\right\}+\frac{\mathrm{i}}{\alpha\,Re}\left\{\bar{\mu}\left(\hat{u}'''-\alpha^2\hat{u}'+\alpha^2\frac{\hat{v}''}{\mathrm{i}\alpha}-\alpha^4\frac{\hat{v}}{\mathrm{i}\alpha}\right)\right.$$

$$+2\bar{\mu}'(\hat{u}''-\alpha^2\hat{u})+\bar{\mu}''\left(\hat{u}'-\alpha^2\frac{\hat{v}}{\mathrm{i}\alpha}\right)+\hat{\mu}(\bar{u}'''+\alpha^2\bar{u}')+2\hat{\mu}'\bar{u}''+\hat{\mu}''\bar{u}'\Big\}=0, \tag{2.4}$$

$$\bar{\rho}\bar{c}_p\left\{(\bar{u}-\hat{c})\,\hat{\theta}+\bar{\theta}'\frac{\hat{v}}{\mathrm{i}\alpha}\right\}+\frac{\mathrm{i}}{\alpha Re\,Pr}\{\bar{k}(\hat{\theta}''-\alpha^2\hat{\theta})+\bar{k}'\hat{\theta}'+\hat{k}\bar{\theta}''+\hat{k}'\bar{\theta}'\}$$

$$+K_{\rho T}\,Ec\left\{\bar{\beta}(1+\epsilon\bar{\theta})\,(\bar{u}-\hat{c})\,\hat{p}-\frac{\mathrm{i}}{\alpha}\frac{\mathrm{d}\bar{p}}{\mathrm{d}x}[\bar{\beta}(1+\epsilon\bar{\theta})\,\hat{u}+(\hat{\beta}(1+\epsilon\bar{\theta})+\bar{\beta}\epsilon\hat{\theta})\,\bar{u}]\right\}$$

$$-\frac{Ec}{Re}\left\{2\bar{\mu}\bar{u}'\,\hat{v}-\frac{\mathrm{i}}{\alpha}(2\bar{\mu}\bar{u}'\,\hat{u}'+\hat{\mu}\bar{u}'^2)\right\}=0, \tag{2.5}$$

with $$Re=\frac{\rho^*_{\mathrm{R}}U^*_{\mathrm{R}}L^*_{\mathrm{R}}}{\mu^*_{\mathrm{R}}},\quad Fr=\frac{U^*_{\mathrm{R}}}{(g^*L^*_{\mathrm{R}})^{\frac{1}{2}}},\quad Pr=\frac{\mu^*_{\mathrm{R}}c^*_{p\mathrm{R}}}{k^*_{\mathrm{R}}},\quad Ec=\frac{U^{*2}_{\mathrm{R}}}{c^*_{p\mathrm{R}}\Delta T^*_{\mathrm{R}}}.$$

which are, respectively, the Reynolds, Froude, Prandtl and Eckert numbers. As usual, quadratic terms are neglected (linear stability theory) and mean-flow quantities are assumed to be only y-dependent (parallel-flow assumption). The notation a' denotes the derivative of a with respect to y.

The associated boundary conditions for the boundary-layer mean flow are

$$y=0:\quad \hat{u}=\hat{v}=\hat{\theta}=0, \tag{2.6a}$$

$$y\to\infty:\quad \hat{u}=\hat{v}=\hat{\theta}=0. \tag{2.6b}$$

The thermal expansion coefficient β^* in (2.5), associated with the work done by compression, is $\beta^*=-(\partial\rho^*/\partial T^*)/\rho^*$ with $\beta=\beta^*T^*_{\mathrm{R}}$. It is formally treated as extra property, but is physically related to the density ρ^*. The quantities $K_{\rho T}$ and ϵ in the compression work terms are defined in (3.2) below. For constant density β^* is zero. In (2.4), σ is an angle measured from a line perpendicular to g^*. All equations are non-dimensionalized with respect to a reference state R. The non-dimensional temperature of the mean flow is $\bar{\theta}=(\bar{T}^*-T^*_{\mathrm{R}})/\Delta T^*_{\mathrm{R}}$; the amplitude function is $\hat{\theta}=\hat{T}^*/\Delta T^*_{\mathrm{R}}$. Here ΔT^*_{R} denotes a characteristic temperature difference of the flow field. For constant properties, $\bar{\rho}=\bar{\mu}=\bar{k}=\bar{c}_p=1$ and $\hat{\rho}=\hat{\mu}=\hat{k}=\hat{c}_p=0$.

When the viscosity varies but all other properties are constant, the continuity equation (2.3) reduces to $\hat{u}+\hat{v}'/\mathrm{i}\alpha=0$. In this case a stream function $\hat{\phi}$ can be introduced by

$$\hat{u}=\hat{\phi}',\quad \hat{v}=-\mathrm{i}\alpha\hat{\phi}. \tag{2.7}$$

For later use, the basic equations are also given for this case. Neglecting viscous dissipation effects ($Ec\to 0$ asymptotically), the basic equations for variable viscosity are

$$(\bar{u}-\hat{c})\,(\hat{\phi}''-\alpha^2\hat{\phi})-\bar{u}''\hat{\phi}+\frac{\mathrm{i}}{\alpha Re}\{\bar{\mu}(\hat{\phi}^{\mathrm{iv}}-2\alpha^2\hat{\phi}''+\alpha^4\hat{\phi})+2\bar{\mu}'(\hat{\phi}'''-\alpha^2\hat{\phi}')$$

$$+\bar{\mu}''(\hat{\phi}''+\alpha^2\hat{\phi})+\hat{\mu}(\bar{u}'''+\alpha^2\bar{u}')+2\hat{\mu}'\bar{u}''+\hat{\mu}''\bar{u}'\}=0, \tag{2.8}$$

$$(\bar{u}-\hat{c})\,\hat{\theta}-\bar{\theta}'\hat{\phi}+\frac{\mathrm{i}}{\alpha Re\,Pr}(\hat{\theta}''-\alpha^2\hat{\theta})=0. \tag{2.9}$$

Equation (2.8) is identical to the equation derived by Wazzan *et al.* (1972) when $\hat{\mu}=0$ is assumed. They neglect temperature fluctuations completely (but without giving a justification). The boundary conditions for $\hat{\phi}$, $\hat{\phi}'$ and $\hat{\theta}$ are zero for $y=0$ and $y\to\infty$ according to (2.6).

3. Perturbation procedure

With a^* representing one of the physical properties ρ^*, μ^*, k^* and c_p^*, the Taylor series expansion reads

$$a = \frac{a^*}{a_R^*} = 1 + \epsilon K_{aT}\,\theta + \tilde{\epsilon} K_{ap}\,p + O(\epsilon^2, \tilde{\epsilon}^2, \epsilon\tilde{\epsilon}), \tag{3.1}$$

with $$\epsilon = \frac{\Delta T_R^*}{T_R^*}, \quad \tilde{\epsilon} = \frac{\rho_R^* U_R^{*2}}{p_R^*}, \quad K_{aT} = \left[\frac{\partial a^*}{\partial T^*}\frac{T^*}{a^*}\right]_R, \quad K_{ap} = \left[\frac{\partial a^*}{\partial p^*}\frac{p^*}{a^*}\right]_R. \tag{3.2}$$

Here ϵ and $\tilde{\epsilon}$ are introduced as small (perturbation) parameters. The Taylor series are truncated after the linear terms. When the series are continued to higher orders, additional K_a-values appear which contain second and mixed derivatives for the next higher order. K_{aT} and K_{ap} are properties of the fluid (see e.g. table 1).

For most fluids, the pressure dependence, K_{ap}, is much smaller than the temperature dependence, K_{aT}. An important exception in this respect is the density of gases. For a perfect gas, for example, $K_{\rho T} = -K_{\rho p} = 1$, so that the pressure dependence cannot always be neglected. Since $\tilde{\epsilon} K_{\rho p} = \kappa Ma^2$ (where $\kappa = c_p^*/c_v^*$), the pressure dependence of gases can be neglected for small Mach numbers only (for details see Herwig 1987).

In a perturbation solution of the stability problem based on (3.1), all variable-property effects are independent. There is an ascending order of accuracy depending on how many terms of the expansion (3.1) are considered. When only linear terms are taken into account, we will call this a *linear perturbation theory*. What follows is a linear analysis, but extension to higher orders with respect to ϵ and $\tilde{\epsilon}$ is straightforward.

Owing to the decomposition $a = \bar{a} + \hat{a}\exp[\mathrm{i}\alpha(x - \hat{c}t)]$, equation (3.1) reads

$$\bar{a} = 1 + \epsilon K_{aT}\,\bar{\theta} + \tilde{\epsilon} K_{ap}\,\bar{p} + O(\epsilon^2, \tilde{\epsilon}^2, \epsilon\tilde{\epsilon}) \tag{3.3a}$$

$$\hat{a} = \epsilon K_{aT}\,\hat{\theta} + \tilde{\epsilon} K_{ap}\,\hat{p} + O(\epsilon^2, \tilde{\epsilon}^2, \epsilon\tilde{\epsilon}). \tag{3.3b}$$

The mean flow field is affected by variable-property effects through $\bar{\rho}$, $\bar{\mu}$, $\bar{k}$ and $\bar{c}_p$, whereas the stability equations (2.3)–(2.5) are affected by the mean as well as by the disturbance parts of the properties.

Equation (3.3) suggests an expansion of all mean flow and disturbance quantities of the general form:

$$a = a_0 + \epsilon(K_{\rho T}\,a_{1\rho} + K_{\mu T}\,a_{1\mu} + K_{kT}\,a_{1k} + K_{cT}\,a_{1c}) + \tilde{\epsilon}(K_{\rho p}\,\tilde{a}_{1\rho} + K_{\mu p}\,\tilde{a}_{1\mu} + K_{kp}\,\tilde{a}_{1k} + K_{cp}\,\tilde{a}_{1c}) + O(\epsilon^2, \tilde{\epsilon}^2, \epsilon\tilde{\epsilon}) \tag{3.4}$$

In (3.4) a represents: $\bar{u}$, $\hat{u}$, $\bar{v}$, $\hat{v}$, $\bar{p}$, $\hat{p}$, $\bar{\theta}$, $\hat{\theta}$, $\hat{\phi}$, $\hat{c}$.

When only the temperature dependence of viscosity is assumed to be important, with all other property variations neglected, the mean flow quantities and amplitude functions from (3.4) read

$$\bar{u} = \bar{u}_0 + \epsilon K_{\mu T}\,\bar{u}_{1\mu} + O(\epsilon^2), \quad \bar{\theta} = \bar{\theta}_0 + \epsilon K_{\mu T}\,\bar{\theta}_{1\mu} + O(\epsilon^2); \tag{3.5}$$

$$\hat{\phi} = \hat{\phi}_0 + \epsilon K_{\mu T}\,\hat{\phi}_{1\mu} + O(\epsilon^2), \quad \hat{\theta} = \hat{\theta}_0 + \epsilon K_{\mu T}\,\hat{\theta}_{1\mu} + O(\epsilon^2). \tag{3.6}$$

In this special case, the complex parameter $\hat{c}$ is

$$\hat{c} = \hat{c}_0 + \epsilon K_{\mu T}\,\hat{c}_{1\mu} + O(\epsilon^2) \tag{3.7}$$

with the amplification rate $c_i = c_{0i} + \epsilon K_\mu\, c_{1\mu i}$. A crucial step in the theory is the

Temperature dependence		Pressure dependence	
$K_{\rho T}$	-0.057	$K_{\rho p}$	0.00005
$K_{\mu T}$	-7.132	$K_{\mu p}$	-0.00025
K_{kT}	0.823	K_{kp}	0.00008
K_{cT}	-0.052	K_{cp}	-0.00006

TABLE 1. K_{aT} and K_{ap} for water at $T^*_R = 293\,\mathrm{K}$, $p^*_R = 1$ bar; $a \hat{=} \rho, \mu, k, c_p$

expansion of the parameter $\hat{c}$ in the same way as the expansion of the functions $\bar{u}$, $\bar{\theta}$, $\hat{\phi}$ and $\hat{\theta}$. This leads to the specific form of first-order equation (4.9) below, from which $\hat{c}_{1\mu}$ can be determined.

Inserting the expansions (3.3)–(3.7) into (2.3)–(2.5) and collecting terms with respect to $\epsilon K_{\rho T}$, $\epsilon K_{\mu T}$ etc., gives the asymptotic equations for the property influence.

4. Special case: temperature-dependent viscosity, flat-plate boundary-layer flow

From the expansions (3.4) it is obvious that effects of different physical properties are independent. Within linear theory with respect to ϵ and $\tilde{\epsilon}$ there are no mixed terms of temperature and pressure dependence. Therefore, a single property effect will be selected to serve as an example. All other effects can be handled similarly and added without changing the previous solution. Each effect is a function of the Prandtl number only.

A theory with temperature-dependent viscosity with all other property variations neglected is a good approximation for water, since the magnitude of $K_{\mu T}$ is considerably higher than that of all other K_{aT} and K_{ap} (see table 1). As an example, the flat-plate boundary layer with a wall at constant temperature $T^*_w \neq T^*_R$ was solved. The reference temperature T^*_R is the temperature T^*_∞ far away from the wall. The dimensionless temperature is $\theta = (T^* - T^*_\infty)/(T^*_w - T^*_\infty)$, i.e. $\Delta T_R = T^*_w - T^*_\infty$. For other boundary layers (with pressure gradients) only the mean flow quantities must be changed.

4.1. *Mean flow quantities*

The differential equations for a flat-plate flow with temperature-dependent viscosity but viscous heating neglected, i.e. $Ec \rightarrow 0$, are (Gersten & Herwig 1984)

$$(\bar{\mu} f'')' + ff'' = 0, \tag{4.1}$$

$$\bar{\theta}'' + Pr\, f\bar{\theta}' = 0, \tag{4.2}$$

with the boundary conditions

$$y = 0: \quad f = f' = \bar{\theta} - 1 = 0; \qquad y \rightarrow \infty: \quad f' - 1 = \bar{\theta} = 0. \tag{4.3}$$

Here, $f(y)$ is the self-similar stream function, $y = y^*/L^*_R$ is the similarity variable, which arises in the usually defined way when L^*_R is set equal to $[2\eta^*_R x^*/(\rho^*_R U^*_R)]^{\frac{1}{2}}$, with $U^*_R = U^*_\infty$ (velocity outside the boundary layer). The mean velocity is $\bar{u} = f'$.

Inserting the expansions $\bar{\mu} = 1 + \epsilon K_{\mu T}\bar{\theta} + O(\epsilon^2)$ according to (3.3a), and $f = f_0 + \epsilon K_{\mu T} f_{1\mu} + O(\epsilon^2)$, $\bar{\theta} = \bar{\theta}_0 + O(\epsilon)$ according to (3.4), with f expanded like $\bar{u}$, into (4.1) and (4.2) gives

$$f_0''' + f_0 f_0'' = 0, \quad \bar{\theta}_0'' + Pr\, f_0 \bar{\theta}_0' = 0, \tag{4.4}$$

$$f_{1\mu}''' + f_0 f_{1\mu}'' + f_0'' f_{1\mu} = -(f_0''\bar{\theta}_0)', \tag{4.5}$$

with the associated boundary conditions from (4.3). For this case, the first-order temperature function $\bar{\theta}_{1\mu}$ will not be needed in the linear perturbation theory.

Solutions $f_0' = \bar{u}_0$, $\bar{\theta}_0(Pr)$ and $f_{1\mu}'(Pr) = \bar{u}_{1\mu}(Pr)$ can be found by a standard Runge–Kutta integration (see e.g. Gersten & Herwig 1984).

4.2. *Amplitude functions*

The stability equations (2.8) and (2.9), from which the amplitude functions $\hat{\phi}$ and $\hat{\theta}$ can be determined, are now subject to a perturbation procedure similar to that of the mean flow. Now $\hat{\phi}$, $\hat{\theta}$ and $\hat{c}$ are also expanded as previously indicated in (3.6) and (3.7). The amplitude function $\hat{\mu}$ according to (3.3*b*) is

$$\hat{\mu} = \epsilon K_{\mu T}\hat{\theta}_0 + O(\epsilon^2). \tag{4.6}$$

Inserting all the expansions into (2.8) and (2.9), and collecting terms of equal magnitude with respect to $\epsilon K_{\mu T}$, leads to the following set of stability equations:

$$(\bar{u}_0-\hat{c}_0)(\hat{\phi}_0''-\alpha^2\hat{\phi}_0)-\bar{u}_0''\hat{\phi}_0+\frac{\mathrm{i}}{\alpha Re}(\hat{\phi}_0^{\mathrm{iv}}-2\alpha^2\hat{\phi}_0''+\alpha^4\hat{\phi}_0) = 0, \tag{4.7}$$

$$(\bar{u}_0-\hat{c}_0)\hat{\theta}_0+\frac{\mathrm{i}}{\alpha Re\, Pr}(\hat{\theta}_0''-\alpha^2\hat{\theta}_0) = \bar{\theta}_0'\hat{\phi}_0, \tag{4.8}$$

$$\begin{aligned}(\bar{u}_0-\hat{c}_0)(\hat{\phi}_{1\mu}''-\alpha^2\hat{\phi}_{1\mu})-\bar{u}_0''\hat{\phi}_{1\mu}+\frac{\mathrm{i}}{\alpha Re}(\hat{\phi}_{1\mu}^{\mathrm{iv}}-2\alpha^2\hat{\phi}_{1\mu}''+\alpha^4\hat{\phi}_{1\mu})\\ = -(\bar{u}_{1\mu}-\hat{c}_{1\mu})(\hat{\phi}_0''-\alpha^2\hat{\phi}_0)+\bar{u}_{1\mu}''\hat{\phi}_0-\frac{\mathrm{i}}{\alpha Re}[\bar{\theta}_0(\hat{\phi}_0^{\mathrm{iv}}-2\alpha^2\hat{\phi}_0''+\alpha^4\hat{\phi}_0)\\ +2\bar{\theta}_0'(\hat{\phi}_0'''-\alpha^2\hat{\phi}_0')+\bar{\theta}_0''(\hat{\phi}_0''+\alpha^2\hat{\phi}_0)+\hat{\theta}_0(\bar{u}_0'''+\alpha^2\bar{u}_0')+2\hat{\theta}_0'\bar{u}_0''+\hat{\theta}_0''\bar{u}_0'],\end{aligned} \tag{4.9}$$

with the associated boundary conditions

$$y = 0: \quad \hat{\phi}_0 = \hat{\phi}_0' = \hat{\phi}_{1\mu} = \hat{\phi}_{1\mu}' = \hat{\theta}_0 = 0, \tag{4.10a}$$

$$y = \infty: \quad \hat{\phi}_0 = \hat{\phi}_0' = \hat{\phi}_{1\mu} = \hat{\phi}_{1\mu}' = \hat{\theta}_0 = 0. \tag{4.10b}$$

Equation (4.7) is the classical OS equation for constant properties which is the zero-order equation of the asymptotic expansion with respect to ϵ. It is well-known that the OS equation describes an eigenvalue problem which owing to its stiffness is difficult to solve numerically (see e.g. Mack 1984).

With these difficulties in mind, the mathematical nature of (4.7), (4.8) and (4.9) as well as a numerical solution procedure will be discussed in the next section.

5. Numerical solutions

As far as the classical OS equation is concerned, there are basically three different types of numerical solution procedures that have been applied successfully: shooting methods, finite-difference implicit solution methods, and spectral methods. In this investigation shooting methods were employed.

5.1. *Zero-order momentum (classical OS equation)*

Equation (4.7) for $y \rightarrow \infty$ reduces to a fourth-order linear differential equation with constant coefficients, since $\bar{u}_0 - 1 = \bar{u}_0'' = 0$ at the outer edge of the boundary layer. Two of the four fundamental solutions of this reduced equation ($\hat{\phi}_{01} = \exp[-\alpha y]$, $\hat{\phi}_{03} = \exp[-\hat{\gamma}y]$, $\hat{\gamma}^2 = \alpha^2 + \mathrm{i}\alpha Re(1-\hat{c}_0)$) are left as non-zero solutions, when the

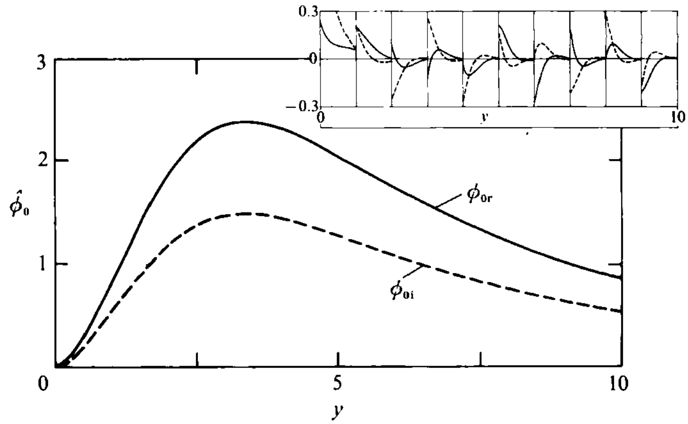

FIGURE 1. Zero-order amplitude function $\hat{\phi}_0$: $Re = 519.1$, $c_{0i} = 0$ ($\alpha = 0.3034$, $c_{0r} = 0.3965$). Inserted picture shows ten subregions of orthonormalization with the two functions ϕ_{03r} (—) and ϕ_{03i} (----). Increasing the number of subregions will increase the computational accuracy.

boundary conditions (4.10*b*) are applied. These two solutions are then integrated through the boundary layer. At the wall, the boundary conditions (4.10*a*) are satisfied when

$$\hat{\boldsymbol{D}} = \begin{vmatrix} \hat{\phi}_{01} & \hat{\phi}_{03} \\ \hat{\phi}'_{01} & \hat{\phi}'_{03} \end{vmatrix} = 0 + \mathrm{i}0. \tag{5.1}$$

For example, the complex determinant $\hat{\boldsymbol{D}}$ will be zero only for specific values of the complex eigenvalue $\hat{c}_0$, when α and Re are fixed.

Owing to the stiffness of (4.7) the integration is performed by applying the so-called Gram–Schmidt orthonormalization. Defining a four-component function space

$$\hat{\boldsymbol{\Phi}}(y) = (\hat{\phi}(y), \hat{\phi}'(y), \hat{\phi}''(y), \hat{\phi}'''(y)), \tag{5.2}$$

the orthonormalization is a linear combination of the original vectors $\hat{\boldsymbol{\Phi}}_{01}$ and $\hat{\boldsymbol{\Phi}}_{03}$, which are replaced by

$$\hat{\boldsymbol{\Phi}}^{\mathrm{o}}_{01} = \frac{\hat{\boldsymbol{\Phi}}_{01}}{|\hat{\boldsymbol{\Phi}}_{01}|}, \quad \hat{\boldsymbol{\Phi}}^{\mathrm{o}}_{03} = \frac{\hat{\boldsymbol{\Phi}}_{03} - (\hat{\boldsymbol{\Phi}}_{03}\hat{\bar{\boldsymbol{\Phi}}}^{\mathrm{o}}_{01})\,\hat{\boldsymbol{\Phi}}^{\mathrm{o}}_{01}}{|\hat{\boldsymbol{\Phi}}_{03} - (\hat{\boldsymbol{\Phi}}_{03}\hat{\bar{\boldsymbol{\Phi}}}^{\mathrm{o}}_{01})\,\hat{\boldsymbol{\Phi}}_{01}|}. \tag{5.3}$$

Here, $\hat{\bar{\boldsymbol{\Phi}}}$ refers to the complex-conjugate vector. The magnitude of $\hat{\boldsymbol{\Phi}}$ is given by $|\hat{\boldsymbol{\Phi}}| = (\hat{\boldsymbol{\Phi}}\hat{\bar{\boldsymbol{\Phi}}})^{\frac{1}{2}}$. This allows the eigenvalues to be computed without significant round-off errors. For more details, see for example Mack (1976).

From our calculations, the critical Reynolds number is $Re_c = 519.1$. This agrees very well with the results of other studies (Schlichting 1982: $Re_c = 520$; Kümmerer 1973: $Re_c = 521$).

5.2. *Zero-order temperature*

The thermal energy stability equation (4.8) is solved to determine the amplitude function $\hat{\theta}_0(y)$. This can be done only when $\hat{\phi}_0(y)$ is known from the solution of (4.7). The solution procedure used in §5.1 provides $\phi_0(y)$ only as piecewise steady functions in subregions of $0 \leqslant y \leqslant y_e$ ($y_e \,\hat{=}$ outer edge of the boundary layer) as a consequence of the Gram–Schmidt orthonormalization. Thus, as a first step, $\hat{\phi}_0(y)$ must be found as a continuous function in $0 \leqslant y \leqslant y_e$ (see figure 1).

Starting from the wall, the stored results of $\hat{\phi}_{01}$ and $\hat{\phi}_{03}$ in the subregions of orthonormalization must be linearly combined in such a way that at the intersections of the subregions continuous functions of $\hat{\boldsymbol{\Phi}}_0 = (\hat{\phi}_0, \hat{\phi}'_0, \hat{\phi}''_0, \hat{\phi}'''_0)$ result. This is achieved since $\hat{\boldsymbol{\Phi}}^o_{01}$ and $\hat{\boldsymbol{\Phi}}^o_{03}$ in each subregion were determined by an integration based on (5.3). This 'patching procedure' can provide $\hat{\boldsymbol{\Phi}}_0$ as the continuous function needed in (4.8), see figure 1. Since $\hat{\phi}_0$ can be determined from (4.7) only up to a (complex) constant, $\hat{\phi}_0$ can be arbitrarily normalized. In figure 1 we have set $\max(\hat{\phi}'_0) = (1+\mathrm{i}1)$ for normalization.

The thermal energy equation (4.8) for $\hat{\theta}_0(y)$ is a non-homogeneous linear second-order differential equation with zero boundary conditions. It is also a stiff differential equation like (4.7), and was solved by the so-called multiple shooting method (see Stoer & Bulirsch 1978). In this method, the whole solution domain is cast into subregions. Then a first step integration is performed starting from assumed boundary conditions in each subregion (taking into account the boundary conditions at the wall and for $y \to \infty$). In subsequent steps, the discontinuities at the boundaries of the subregions are removed so that a continuous function $\hat{\theta}_0$ results.

The outer boundary condition (4.10*b*) for $\hat{\theta}_0$ can be replaced by

$$y \to \infty: \quad \hat{\theta}'_0 = -(\alpha^2 + \mathrm{i}\alpha Re\, Pr(1-\hat{c}_0))^{\frac{1}{2}}\hat{\theta}_0. \tag{5.4}$$

This alternative boundary condition, which is more convenient for the numerical solution, follows from (4.8) for $y \to \infty$.

In figure 2, the amplitude function $\hat{\theta}_0(y)$ is shown for a specific set of parameters $(Pr, Re, c_{0\mathrm{i}})$. In these curves, the y-position of the critical layer (i.e. the position where $\bar{u}_0 = c_{0\mathrm{r}}$) is marked by an arrow. At this position, the stability equations become singular for $1/Re = 0$ as can be seen in (2.8). For large but finite Reynolds numbers substantial changes may occur in the vicinity of this layer. This does not pose problems for the numerical solution for Reynolds numbers considered in this study. (Note that this layer has no special effect at all on the amplitude function $\hat{\phi}_0$ for a Reynolds number as low as in figure 1!)

5.3. *First-order momentum*

Equation (4.9), as well as all higher-order momentum equations (which are not considered here explicitly), is a non-homogeneous differential equation of the general form

$$\mathrm{L}[\hat{\phi}_{i\mu}, \hat{c}_0] = f(\hat{\phi}_j, \hat{\theta}_j, \hat{c}_{i\mu}); \quad i > j, j = 0, \ldots, \tag{5.5}$$

with the OS differential operator L given by

$$\mathrm{L}[\hat{\phi}, \hat{c}_0] = (\bar{u}_0 - \hat{c}_0)(\hat{\phi}'' - \alpha^2\hat{\phi}) - \bar{u}''_0\hat{\phi} + \frac{\mathrm{i}}{\alpha Re}(\hat{\phi}^{\mathrm{iv}} - 2\alpha^2\hat{\phi}'' + \alpha^4\hat{\phi}). \tag{5.6}$$

Specific values of $\hat{c}_{i\mu}$ must be found for which (5.5) has a solution. The term 'eigenvalue' should be used only in connection with homogeneous equations. Therefore, $\hat{c}_{1\mu}$ will be called a 'first-order parameter' from now on. The corresponding solution for the first order will be denoted by $\hat{\phi}_{1\mu\mathrm{p}}$ (p for particular solution). The general solution of the first-order problem then is

$$\hat{\phi}_{1\mu} = \hat{\phi}_{1\mu\mathrm{p}} + \hat{C}\hat{\phi}_0 \tag{5.7}$$

since $\hat{\phi}_0$ satisfies $\mathrm{L}[\hat{\phi}, \hat{c}_0] = 0$. Owing to the undetermined complex constant $\hat{C}$ in (5.7) integration can start from the wall fixing $\hat{\phi}''_{1\mu}(0)$ arbitrarily, for example. Integration was again performed by the multiple shooting method, using $\hat{c}_{1\mu}$ as shooting parameter.

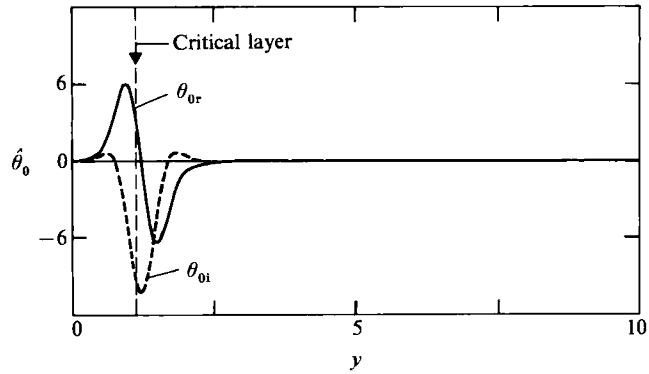

FIGURE 2. Zero-order amplitude function $\hat{\theta}_0$: $Pr = 8.1$, $Re = 519.1$, $c_{0i} = 0$.

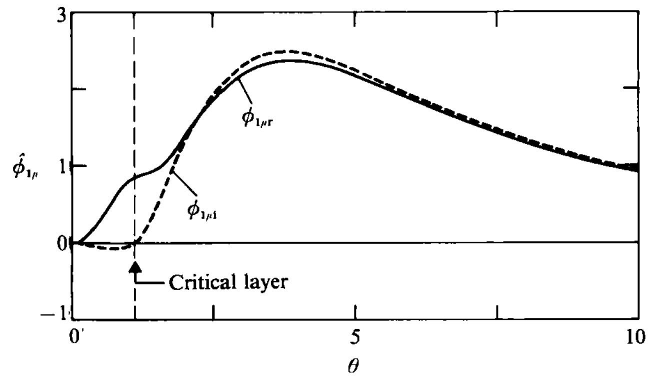

FIGURE 3. First-order amplitude function $\hat{\phi}_{1\mu}$: $Pr = 8.1$, $Re = 519.1$, $c_{0i} = 0$ ($c_{1\mu r} = 0.0242$, $c_{1\mu i} = 0.0923$).

The process of determining the first-order parameter $\hat{c}_{1\mu}$ is a linear process, since $\hat{\phi}_{1\mu}$ and its derivatives are not multiplied by $\hat{c}_{1\mu}$. Therefore, no iteration is needed to determine $\hat{c}_{1\mu}$. This is different from the process of determining $\hat{c}_0$, since $\hat{\phi}_0$ is multiplied by $\hat{c}_0$ in (4.7).

The outer boundary condition (4.10*b*) for $\hat{\phi}_{1\mu}$ and $\hat{\phi}'_{1\mu}$, respectively, can again be replaced by an alternative form which is more suitable for numerical solutions in a finite region $0 < y < y_e$. For $y \to \infty$, the right-hand side of (4.9) is $\hat{c}_{1\mu}(\hat{\phi}''_0 - \alpha^2\hat{\phi}_0)$. But, for $y \to \infty$, $\hat{\phi}_0 = \hat{C}_1\hat{\phi}_{01} + \hat{C}_3\hat{\phi}_{03}$, see §5.1, with $\hat{\phi}_{01} = \exp[-\alpha y]$, $\hat{\phi}_{03} = \exp[-(\alpha^2 + \mathrm{i}\alpha Re(1-\hat{c}_0))^{\frac{1}{2}}y]$, so that $\hat{\phi}_0$ is dominated by $\hat{\phi}_{01}$ for $y \to \infty$. With $\hat{\phi}_0 = \hat{C}_1\hat{\phi}_{01}$ the term $\hat{c}_{1\mu}(\hat{\phi}''_0 - \alpha^2\hat{\phi}_0)$ is zero, so that the whole right-hand side of (4.9) is zero for $y \to \infty$. As a consequence

$$\hat{\phi}_{1\mu} = \hat{C}_1\hat{\phi}_{01} + \hat{C}_3\hat{\phi}_{03} \quad \text{for} \quad y \to \infty \tag{5.8}$$

holds so that (4.10*b*) can be replaced by

$$\hat{\phi}''_{1\mu} + \alpha\hat{\phi}'_{1\mu} = -\hat{\gamma}(\hat{\phi}'_{1\mu} + \alpha\hat{\phi}_{1\mu}), \quad \hat{\phi}'''_{1\mu} + \alpha\hat{\phi}''_{1\mu} = \hat{\gamma}^2(\hat{\phi}'_{1\mu} + \alpha\hat{\phi}_{1\mu}). \tag{5.9}$$

In figure 3 the amplitude function $\hat{\phi}_{1\mu}(y)$ is shown for the same parameters (Pr, Re, c_{0i}) as in figure 2. For normalization we have set $\hat{\phi}''_{1\mu}(0) = (1 + \mathrm{i}1)$. The constant $\hat{c}_{1\mu}$ for this case is $\hat{c}_{1\mu} = 0.0242 + \mathrm{i}0.0923$.

The influence of variable viscosity on the critical Reynolds number can be determined from figures 4 (*a*)–4 (*c*). In these figures the eigenvalue c_{0i} and the first-

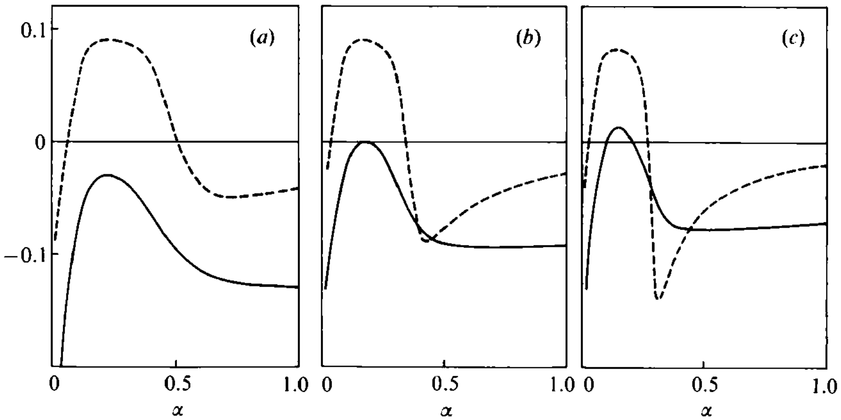

FIGURE 4. Eigenvalues c_{0i} (—) and first-order parameters $c_{1\mu i}$ (----), $Pr = 8.1$ for three different Reynolds numbers ($c_i = c_{0i} + \epsilon K_{\mu T} c_{1\mu i}$): (*a*) $Re = 200$; (*b*) $Re = 519.1$; (*c*) $Re = 1000$.

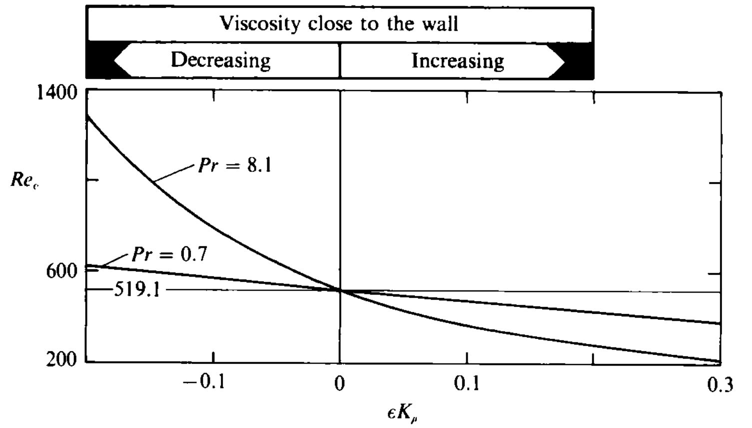

FIGURE 5. Critical Reynolds number for non-isothermal boundary-layer flows of fluids with temperature-dependent viscosity; $\epsilon = (T^*_w - T^*_\infty)/T^*_\infty$; $K_{\mu T} = [(\partial\mu^*/\partial T^*)\,(T^*/\mu^*)]_\infty$.

order parameter $c_{1\mu i}$ are given as functions for α for three different Reynolds numbers. The full line is the eigenvalue c_{0i} of the classical Orr–Sommerfeld problem. As the Reynolds number increases, flow instabilities occur as indicated by the positive values of c_i. For constant properties, this occurs for $Re_{c0} = 519.1$ for the first time (see figure 4*b*). For variable properties, the critical Reynolds number Re_c is reached when $c_{0i} + \epsilon K_{\mu T} c_{1\mu T} = 0$. From figures 4(*a*) and 4(*c*), for example, we find that $Re_c = 200$ for $\epsilon K_{\mu T} = 0.3412$ and $Re_c = 1000$ for $\epsilon K_{\mu T} = -0.1498$, since then $c_{0i} + \epsilon K_{\mu T} c_{1\mu i} = 0$ for just one α.

Based on a large number of curves for $c_{0i}(\alpha)$ and $c_{1\mu i}(\alpha)$ like those in figure 4, we can find the critical Reynolds number Re_c as a function of the perturbation parameter $\epsilon K_{\mu T}$ ($K_{\mu T}$ being an $O(1)$ constant with respect to the asymptotic expansion). In figure 5, these curves are depicted for two different Prandtl numbers. From this figure we can conclude that the flow is stabilized when $\epsilon K_{\mu T} < 0$, since then the critical Reynolds number lies above that for the isothermal (constant-property) case. Physically this corresponds to a decrease of viscosity (compared to the reference viscosity μ^*_∞) within the boundary layer, being strongest at the wall. This result is unexpected since a unique decrease of viscosity, i.e. constant-property flow with a lower viscosity, destabilizes the flow (then $Re_c = [\rho^*_\infty U^*_\infty L^*_R/\mu^*_\infty]_c = 520$ holds further on, so that decreasing μ^*_∞ can increase the actual Reynolds number Re above Re_c). From this we can conclude that the viscosity distribution across the boundary layer is an essential feature for stability characteristics of the flow.

Based on these findings, a reference temperature concept, widely used with variable property flows, is not justified. In this concept, constant-property results are retained for variable-property flows but with all properties in the final results evaluated at a reference temperature T^*_r with $T^*_r = T^*_\infty + j(T^*_w - T^*_\infty)$, $0 < j < 1$ (often $j = \frac{1}{2}$ is expected). For flat-plate boundary-layer flow, however, j would be negative. This again shows that the stability behaviour of constant- and variable-property flows are distinctly different.

The combination $\epsilon K_{\mu T}$ is negative

(*a*) for fluids with $K_{\mu T} < 0$ that are heated ($\epsilon > 0$) (like water with $K_{\mu T} = -7.132$ at 293 K, 1 bar, according to table 1).

(*b*) for fluids with $K_{\mu T} > 0$ that are cooled ($\epsilon < 0$) (like air with $K_{\mu T} = 0.733$ at 293 K, 1 bar, according to Gersten & Herwig 1984).

From figure 5 it can also be concluded that the stabilizing/destabilizing effect for a certain amount of heating (fixed $\epsilon > 0$) for water is much stronger than for air. For example, the deviation of Re_c from $Re_{c0} = 519.1$ is stronger for higher Prandtl numbers and the magnitude of $K_{\mu T}$ is nearly ten times larger for water than for air.

6. Discussion

As mentioned in §3 a crucial step of the whole procedure is to expand the parameter $\hat{c}$ as $\hat{c}_0 + \epsilon K_{\mu T}\hat{c}_1 + O(\epsilon^2)$. All other studies that account for the influence of variable properties still solve eigenvalue problems, i.e. they still have homogeneous differential equations. In the present approach $\hat{c}_1$ must be determined from the non-homogeneous differential equation (4.9).

For a direct comparison of these two approaches, (2.8) was solved directly, i.e. without expansion with respect to ϵ. For this purpose we set $\bar{\mu} = \exp[b(T^* - T^*_\infty)/T^*_\infty]$ and $\hat{\mu} = 0$, i.e. we assumed an exponential viscosity law and neglected temperature fluctuations. Based on these assumptions (2.8) was solved for several values of $(T^*_w - T^*_\infty)/T^*_\infty$. Equation (2.8) is a modified version of the OS equation; it is however mathematically still an eigenvalue problem with eigenvalues $\hat{c}$. In figure 6 the full line is the critical Reynolds number over the temperature difference $T^*_w - T^*_\infty$ from solutions of (2.8). The broken line is the asymptotic result from (4.7), (4.8) and (4.9). Deviations occur for increasing temperature difference because, in the asymptotic approach, higher-order terms are neglected. The effects of temperature fluctuations (included in the asymptotic results, but not in the direct solution of (2.8)) are obviously small.

Neglecting temperature fluctuations completely (as for example Wazzan *et al.* 1972 do in their study), cannot be justified from an asymptotic point of view. In this sense, it is an irrational approximation. To determine its influence quantitatively, we calculated the first-order parameter $c_{1\mu i}$ from (4.9) with $\hat{\theta}_0 = 0$. The $c_{1\mu i}$ curves in figure 4(*a*)–4(*c*) are moderately changed by neglecting temperature fluctuations. Figure 7 compares one curve from figure 4(*a*) ($Re = 200$) with the corresponding case of fluctuating temperature. For higher Reynolds numbers the influence of $\hat{\theta}_0$ decreases.

Our final asymptotic results, which have the Prandtl number as the only solution parameter, can be specified for a certain fluid by specifying $K_{\mu T}$ and for a certain

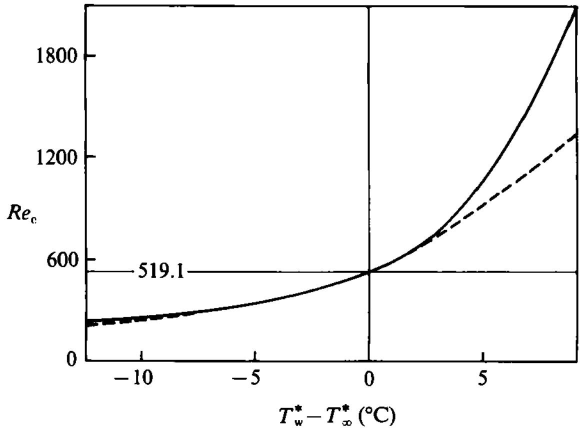

FIGURE 6. Critical Reynolds number for an exponential viscosity law $\bar{\mu} = \exp[b(T^* - T^*_\infty)/T^*_\infty]$; $b = -7.4$; $T^*_\infty = 290\,K$: —, direct solution, (2.8), (no temperature fluctuations); ----, asymptotic solution for $|T^*_w - T^*_\infty|/T^*_\infty \to 0$, (4.7)–(4.9) (with temperature fluctuations).

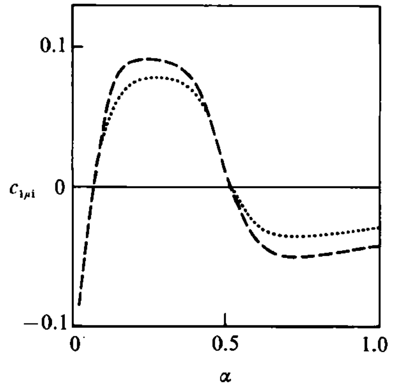

FIGURE 7. First-order parameter $c_{1\mu i}$ with (----) (figure 4a) and without (......) temperature fluctuations: $Pr = 8.1$, $Re = 200$.

temperature difference by specifying ϵ. Results for $Pr = 8.1$ are shown in figure 8, because this is the relevant Prandtl number in the study of Wazzan *et al.* (1972). Their free-stream water temperature was 60 °F which corresponds to $Pr = 8.1$. Whereas our study holds for $\epsilon K_{\mu T} \to 0$ asymptotically, their results are given for particular values of $\epsilon K_{\mu T}$, the lowest of which is $\epsilon K_{\mu T} = -0.43$. (Wazzan *et al.* did not introduce ϵ and $K_{\mu T}$ explicitly; however we inferred from their data what ϵ and $K_{\mu T}$ would be in their study).

Since the exponential viscosity law introduced in connection with figure 6 is that of Wazzan *et al.* (1972) the direct solutions of (2.8) could be extended to higher temperature differences, and thus reach a temperature difference as high as that of the first point in the study by Wazzan *et al.* (1972). This was accomplished by recalculating equation (1) of Wazzan *et al.*, which is equivalent to (2.8) when $\hat{\mu} = 0$. There is a strong increase in the Re_c-slope for increasing heat transfer (due to higher-order effects from an asymptotic analysis). This is indicated by the full curve in figure 8. This increase in slope (and not the linear interpolation which Wazzan *et al.* 1972 assumed) is obviously a characteristic feature of the Re_c-curve for small temperature differences. This kind of behaviour also occurs in a later study by Wazzan, Taghavi & Hsu (1978) for freon-114.

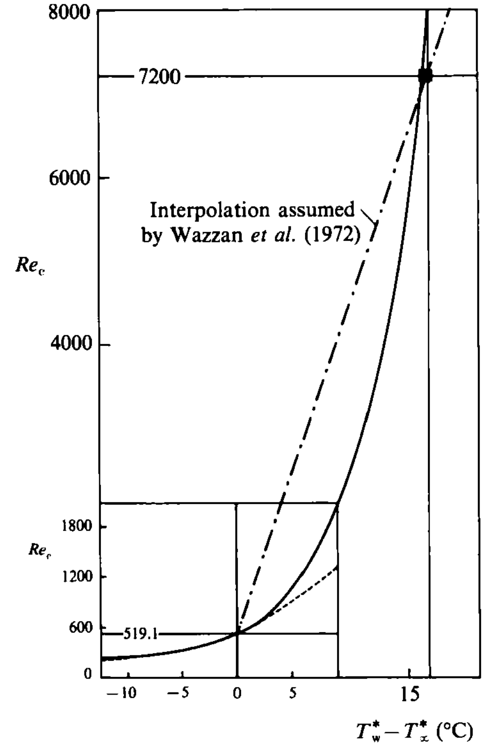

FIGURE 8. Comparing asymptotic and non-asymptotic results: ----, asymptotic results for $|T_w^* - T_\infty^*|/T_\infty^* \rightarrow 0$ (this study, figure 6); ■, first data point from Wazzan *et al.* (1972); —, recalculation of equation (1) in Wazzan *et al.* (1972); equivalent to the direct solution of our equation (2.8) with $\hat{\mu} = 0$.

For all non-zero values of the perturbation parameter the regime for which the first-order results give adequate, quantitative information can differ from case to case. From the specific example of this study we conclude that the stability behaviour for a range of temperature differences for which the critical Reynolds number differs by more than 50% is described with an error of a few percent. In the range of destabilization (left part of figure 6) for example it can be expected that deviations between the first-order asymptotic and the exact results will always be small since the critical Reynolds number must be in the limited range

$$0 < Re_c \leqslant Re_{c\ \text{isothermal}}.$$

7. Conclusions

With a combined asymptotic/numerical method (which might be called the ACFD approach, see Herwig 1990) the influence of variable properties on the stability behaviour of laminar flows can be studied in a systematic and general way. There is no need to specify a particular fluid from the beginning, since the final results in their general form hold for all Newtonian fluids.

For the special case of a flat-plate boundary-layer flow with temperature-dependent viscosity, results show that decreasing the viscosity in the near-wall region stabilizes the flow, whereas linear stability theory predicts that a (global) viscosity decrease would destabilize the flow.

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